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Lax pair for $SU(n)$ Hubbard model

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Abstract

For one dimensional $SU(n)$ Hubbard model, a pair of Lax operators are derived, which give a set of fundamental equations for the quantum inverse scattering method under both periodic and open boundary conditions. This provides another proof of the integrability of the model under periodic boundary condition.

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Integrable strongly correlated electron systems have been an important research subject in condensed matter physics and mathematics. One of the significant models is the 1-D Hubbard. The exact solution was given by Lieb and Wu [1]. However, the integrability was shown twenty years later by Shastry, Olmedilla and Wadati [2, 3]. The integrability and the exact solution of the system under the open boundary condition were discussed by several authors [4, 5]. The Lax pair was first given by Wadati, Olmedilla and Akutsu [6]. Recently, Maassarani and Mathieu have constructed the hamiltonian $SU(n)$ XX model and proved its integrability [7]. Considering two coupled $SU(n)$ XX models, Maassarani succeeded in generalizing Shastry's method to $SU(n)$ Hubbard model [8]. Further, he solved the Yang-Baxter equation to prove the integrability of one dimensional $SU(n)$ Hubbard model [9]. (It is also proved by Martins for $n = 3, 4$ [10].) In this paper, we apply the quantum inverse scattering method to 1-dimensional $SU(n)$ Hubbard model and derive the explicit form of the Lax pair, which gives another proof of the integrability. It is worthy to note that straightforward application of the reflection matrix method to study the problem of the integrable open boundary in the present $SU(n)$ Hubbard model would encounter difficulties due to the fact that the R matrix in reference [9] does not satisfy the crossing symmetry condition and the lack of invertibility of a certain matrix.

The Lax pair formalism will give an effective method for such a system. The 1-dimensional $SU(n)$ model in the Schrödinger picture is given by

$$\mathcal{H} = \sum_{k=1}^N \sum_{\alpha=1}^{n-1} \left(E_{\sigma,k}^{n\alpha} E_{\sigma,k+1}^{\alpha n} + E_{\sigma,k}^{\alpha n} E_{\sigma,k+1}^{n\alpha} + E_{\tau,k}^{n\alpha} E_{\tau,k+1}^{\alpha n} + E_{\tau,k}^{\alpha n} E_{\tau,k+1}^{n\alpha} \right) + \frac{Un^2}{4} \sum_{k=1}^N C_{\sigma,k} C_{\tau,k}, \quad (1)$$

where $E_{a,k}^{\alpha\beta}$ ($a = \sigma, \tau$) is a matrix with zeros everywhere except for a 1 at the intersection of row α and column β :

$$(E^{\alpha\beta})_{lm} = \delta_l^\alpha \delta_m^\beta. \quad (2)$$

The subscripts a, k stand for two different E operators at site k ($k = 1, \dots, N$). The $n \times n$ diagonal matrix C is defined by $C = \sum_{\alpha < n} E^{\alpha\alpha} - E^{nn}$. The Hamiltonian enjoys the $(su(n-1) \oplus u(1))_\sigma \oplus (su(n-1) \oplus u(1))_\tau$ symmetry. The generators are

$$J_a^{\alpha\beta} = \sum_{k=1}^N E_{a,k}^{\alpha\beta}, \quad \text{and} \quad K_a = \sum_{k=1}^N C_{a,k}, \quad \alpha, \beta = 1, \dots, n-1, \quad a = \sigma, \tau.$$

In the rest of this letter we discuss various operators in the Heisenberg picture. When we deal with the operators corresponding to the matrices E and C in the Schrödinger picture, they are denoted by adding a hat ($\hat{}$) to the corresponding matrix:

$$\hat{Q}(t) = e^{i\mathcal{H}t} Q e^{-i\mathcal{H}t}, \quad Q = E \text{ or } C.$$

In the following, we do not indicate the time dependence of the operators. Applying this method to the Hamiltonian (1), we find

$$\begin{aligned}
\frac{d\hat{E}_{a,k}^{nn}}{dt} &= i \sum_{\beta < n} \left(\hat{E}_{a,k}^{\beta n} \hat{E}_{a,k+1}^{n\beta} - \hat{E}_{a,k}^{n\beta} \hat{E}_{a,k+1}^{\beta n} + \hat{E}_{a,k}^{\beta n} \hat{E}_{a,k-1}^{n\beta} - \hat{E}_{a,k}^{n\beta} \hat{E}_{a,k-1}^{\beta n} \right), \\
\frac{d\hat{E}_{a,k}^{n\alpha}}{dt} &= i \sum_{\beta < n} \left(\hat{E}_{a,k}^{\beta\alpha} \hat{E}_{a,k+1}^{n\beta} + \hat{E}_{a,k}^{\beta\alpha} \hat{E}_{a,k-1}^{n\beta} \right) - i \left(\hat{E}_{a,k}^{nn} \hat{E}_{a,k+1}^{n\alpha} + \hat{E}_{a,k}^{nn} \hat{E}_{a,k-1}^{n\alpha} \right) - i \frac{Un^2}{2} \hat{E}_{a,k}^{n\alpha} \hat{C}_{\bar{a},k}, \\
\frac{d\hat{E}_{a,k}^{\alpha n}}{dt} &= i \left(\hat{E}_{a,k}^{nn} \hat{E}_{a,k+1}^{\alpha n} + \hat{E}_{a,k}^{nn} \hat{E}_{a,k-1}^{\alpha n} \right) - i \sum_{\beta < n} \left(\hat{E}_{a,k}^{\alpha\beta} \hat{E}_{a,k+1}^{\beta n} + \hat{E}_{a,k}^{\alpha\beta} \hat{E}_{a,k-1}^{\beta n} \right) + i \frac{Un^2}{2} \hat{E}_{a,k}^{\alpha n} \hat{C}_{\bar{a},k}, \\
\frac{d\hat{E}_{a,k}^{\alpha\beta}}{dt} &= i \left(\hat{E}_{a,k}^{n\beta} \hat{E}_{a,k+1}^{\alpha n} - \hat{E}_{a,k}^{\alpha n} \hat{E}_{a,k+1}^{n\beta} + \hat{E}_{a,k}^{n\beta} \hat{E}_{a,k-1}^{\alpha n} - \hat{E}_{a,k}^{\alpha n} \hat{E}_{a,k-1}^{n\beta} \right), \tag{3}
\end{aligned}$$

where $(a = \sigma, \tau)$ and $(\bar{a} = \tau, \sigma)$. For an infinite system, it is not necessary to specify the boundary condition. However, one should understand $\hat{E}_{a,0}$ is equal to $\hat{E}_{a,N}$ under the periodic boundary condition. In an open boundary system, $\hat{E}_{a,0}$ and $\hat{E}_{a,N+1}$ in the r.h.s. must be regarded as vanishing.

Let us first consider the degenerate case $U = 0$ (the $SU(n)$ XX model [7]). In this case, the equations of motion (3) decouple into two identical sets of equations for σ and τ . Let us introduce the L -operator for each of them:

$$L_{a,k}(\lambda) = \cos(\lambda) S_{a,k}^{(+)} + \sin(\lambda) T_{a,k}^{(+)} + U_{a,k}^{(+)}, \tag{4}$$

in which the blocks $S_{a,k}^{(+)}$, $T_{a,k}^{(+)}$ and $U_{a,k}^{(+)}$ are defined by

$$\begin{aligned}
S_{a,k}^{(+)} &= \left(\sum_{\alpha, \beta < n} \hat{E}_{a,k}^{\alpha\beta} E_{a,au}^{\beta\alpha} \right) + \hat{E}_{a,k}^{nn} E_{a,au}^{nn}, \\
T_{a,k}^{(+)} &= \sum_{\alpha < n} \left(\hat{E}_{a,k}^{nn} E_{a,au}^{\alpha\alpha} + \hat{E}_{a,k}^{\alpha\alpha} E_{a,au}^{nn} \right), \\
U_{a,k}^{(+)} &= \sum_{\alpha < n} \left(\hat{E}_{a,k}^{n\alpha} E_{a,au}^{\alpha n} + \hat{E}_{a,k}^{\alpha n} E_{a,au}^{n\alpha} \right). \tag{5}
\end{aligned}$$

They satisfy various identities inherited from the definition in terms of $E^{\alpha\beta}$'s (2):

$$S_{a,k}^{(+)} T_{a,k}^{(+)} = T_{a,k}^{(+)} S_{a,k}^{(+)} = S_{a,k}^{(+)} U_{a,k}^{(+)} = U_{a,k}^{(+)} S_{a,k}^{(+)} = 0, \quad T_{a,k}^{(+)} U_{a,k}^{(+)} = U_{a,k}^{(+)} T_{a,k}^{(+)} = U_{a,k}^{(+)}, \dots \tag{6}$$

We also introduce the M -operator for each species:

$$\begin{aligned}
M_{a,k}(\lambda) &= \sum_{\beta < n} \left\{ A_1 \hat{E}_{a,k}^{\beta n} \hat{E}_{a,k-1}^{n\beta} + A_2 \hat{E}_{a,k}^{n\beta} \hat{E}_{a,k-1}^{\beta n} \right\} E_{a,au}^{nn} \\
&+ \sum_{\beta < n} B \left\{ (\hat{E}_{a,k}^{n\beta} + \hat{E}_{a,k-1}^{n\beta}) E_{a,au}^{\beta n} + (\hat{E}_{a,k}^{\beta n} + \hat{E}_{a,k-1}^{\beta n}) E_{a,au}^{n\beta} \right\} \\
&+ \sum_{\alpha < n} \left\{ D_1 \hat{E}_{a,k}^{n\alpha} \hat{E}_{a,k-1}^{\alpha n} + D_2 \hat{E}_{a,k}^{\alpha n} \hat{E}_{a,k-1}^{n\alpha} + \sum_{\beta \neq \alpha < n} D_3 (\hat{E}_{a,k}^{\beta n} \hat{E}_{a,k-1}^{n\beta} + \hat{E}_{a,k}^{n\beta} \hat{E}_{a,k-1}^{\beta n}) \right\} E_{a,au}^{\alpha\alpha} \\
&+ \sum_{\alpha < n} \sum_{\beta \neq \alpha < n} F \left\{ \hat{E}_{a,k}^{n\beta} \hat{E}_{a,k-1}^{\alpha n} - \hat{E}_{a,k}^{\alpha n} \hat{E}_{a,k-1}^{n\beta} \right\} E_{a,au}^{\beta\alpha}, \tag{7}
\end{aligned}$$

where A, B, D, F are as yet undetermined functions of a (spectral) parameter λ . The matrices $E_{a,au}$ are the constant matrices with the same definition as $E_{a,k}$, $k = 1, \dots, N$. The subscript au stands for the auxiliary space instead of the quantum space. Thus $L_{a,k}$ and $M_{a,k}$ are $n \times n$ matrices in the auxiliary space. We want to rewrite equations (3) in a matrix Lax pair form

$$\frac{dL_{a,k}(\lambda)}{dt} = M_{a,k+1}(\lambda)L_{a,k}(\lambda) - L_{a,k}(\lambda)M_{a,k}(\lambda). \quad (8)$$

Substituting equations (4) and (7) into the above Lax-pair form (8), we find the solution

$$\begin{aligned} A_1 &= D_1 = i + i \tan(\lambda), & A_2 &= D_2 = i - i \tan(\lambda), \\ D_3 &= i, & B &= -i/\cos(\lambda), & F &= i \tan(\lambda). \end{aligned} \quad (9)$$

From now on we do not denote the λ -dependence of $L_{a,k}$ and $M_{a,k}$ for brevity. The transfer matrix for $SU(n)$ XX model with N sites can be defined by

$$T_{a,XX} = L_{a,N} \cdots L_{a,1},$$

which satisfies

$$\frac{dT_{a,XX}}{dt} = M_{a,N+1}T_{a,XX} - T_{a,XX}M_{a,1}.$$

From this it is standard to show that the trace of $T_{a,XX}$ is independent of time under periodic boundary condition. So the $SU(n)$ XX model is integrable. This Lax-pair is very important for the construction of the Lax-pair of $SU(n)$ Hubbard model. Notice that its integrability was first given by Maassarani and Mathieu in the framework of Yang-Baxter relation [7].

Now, let us consider $U \neq 0$ case. In terms of the above $L_{a,k}$ and $M_{a,k}$, we can rewrite equations (3) as

$$\frac{dL_{a,j}}{dt} = M_{a,j+1}L_{a,j} - L_{a,j}M_{a,j} + i\frac{Un^2}{4}[L_{a,j}, C_{a,au}\hat{C}_{\bar{a},j}], \quad (10)$$

in which the last term in the r.h.s. manifests the coupling between the two species σ and τ . Using the relation $[L_{a,j}, C_{a,au}] = -[L_{a,j}, \hat{C}_{a,j}]$, we obtain

$$\frac{dL_{a,j}}{dt} = M_{a,j+1}L_{a,j} - L_{a,j}M_{a,j} - i\frac{Un^2}{4}[L_{a,j}, \hat{C}_{a,j}\hat{C}_{\bar{a},j}]. \quad (11)$$

We define the following operators for the coupled system

$$\tilde{L}_j = L_{\sigma,j}L_{\tau,j}, \quad \tilde{M}_j = M_{\sigma,j} + M_{\tau,j}, \quad (12)$$

then equation (11) can be written as

$$\frac{d\tilde{L}_j}{dt} = \tilde{M}_{j+1}\tilde{L}_j - \tilde{L}_j\tilde{M}_j - i\frac{Un^2}{4}[\tilde{L}_j, \hat{C}_{\sigma,j}\hat{C}_{\tau,j}]. \quad (13)$$

Now, we want to rewrite the last term in the above equation so that we could obtain the Lax-pair form for the coupled system in a similar form to (8). Following the method given by Wadati et al for $SU(2)$ Hubbard model, we introduce a “rotation” matrix by the $u(1) \otimes u(1)$ charge:

$$I_{au} = \cosh(h/2) + \sinh(h/2)C_{\sigma,au}C_{\tau,au} = \exp\{\frac{h}{2}C_{\sigma,au}C_{\tau,au}\}, \quad (14)$$

where h is a free parameter to be determined later and the “rotated” operators are

$$\mathcal{L}_j = I_{au}\tilde{\mathcal{L}}_jI_{au}, \quad \mathcal{M}_j = I_{au}^{-1}\tilde{\mathcal{M}}_jI_{au}. \quad (15)$$

By this the fundamental blocks $S_{a,k}^{(+)}$, $T_{a,k}^{(+)}$ and $U_{a,k}^{(+)}$ are mapped to their ‘anti-symmetric’ counterparts $S_{a,k}^{(-)}$, $T_{a,k}^{(-)}$ and $U_{a,k}^{(-)}$:

$$\begin{aligned} S_{a,k}^{(-)} &= C_{a,au}S_{a,k}^{(+)} = S_{a,k}^{(+)}C_{a,au} = \left(\sum_{\alpha,\beta < n} \hat{E}_{a,k}^{\alpha\beta}E_{a,au}^{\beta\alpha}\right) - \hat{E}_{a,k}^{nn}E_{a,au}^{nn}, \\ T_{a,k}^{(-)} &= C_{a,au}T_{a,k}^{(+)} = T_{a,k}^{(+)}C_{a,au} = \sum_{\alpha < n} \left(\hat{E}_{a,k}^{nn}E_{a,au}^{\alpha\alpha} - \hat{E}_{a,k}^{\alpha\alpha}E_{a,au}^{nn}\right), \\ U_{a,k}^{(-)} &= C_{a,au}U_{a,k}^{(+)} = -U_{a,k}^{(+)}C_{a,au} = \sum_{\alpha < n} \left(\hat{E}_{a,k}^{n\alpha}E_{a,au}^{\alpha n} - \hat{E}_{a,k}^{\alpha n}E_{a,au}^{n\alpha}\right), \end{aligned} \quad (16)$$

which also satisfy identities similar to those given in (6). Then equation (13) becomes

$$\frac{d\mathcal{L}_j}{dt} = I_{au}^2\mathcal{M}_{j+1}I_{au}^{-2}\mathcal{L}_j - \mathcal{L}_j\mathcal{M}_j - i\frac{Un^2}{4}[\mathcal{L}_j, \hat{C}_{\sigma,j}\hat{C}_{\tau,j}]. \quad (17)$$

Using the definition of I_{au} and \mathcal{M} , we obtain

$$I_{au}^2\mathcal{M}_{j+1}I_{au}^{-2} = \mathcal{M}_{j+1} + Q_{j+1} + Q_j, \quad (18)$$

$$Q_j = -\frac{2i}{\cos(\lambda)}\sinh(h)\left(U_{\sigma,j}^{(-)}C_{\tau,au} + U_{\tau,j}^{(-)}C_{\sigma,au}\right), \quad (19)$$

and

$$\begin{aligned} \frac{d\mathcal{L}_j}{dt} &= (\mathcal{M}_{j+1} + Q_{j+1})\mathcal{L}_j - \mathcal{L}_j(\mathcal{M}_j + Q_j) \\ &\quad + \mathcal{L}_jQ_j + Q_j\mathcal{L}_j - i\frac{Un^2}{4}[\mathcal{L}_j, \hat{C}_{\sigma,j}\hat{C}_{\tau,j}]. \end{aligned} \quad (20)$$

Detailed calculation shows the last line in the above equation to be

$$\begin{aligned} &Q_j\mathcal{L}_j + \mathcal{L}_jQ_j - i\frac{Un^2}{4}[\mathcal{L}_j, \hat{C}_{\sigma,j}\hat{C}_{\tau,j}] \\ &= \left\{\frac{i}{a}\left(b - \frac{1}{a-b}\right)\sinh(2h) + i\frac{Un^2}{4}\frac{a+b}{a-b}\right\}[\mathcal{L}_j, C_{\sigma,au}C_{\tau,au}] \\ &- 2\left\{\frac{i}{a}\frac{a}{a-b}\sinh(2h) - i\frac{Un^2}{4}\frac{2ab}{a-b}\right\}I_{au}\left(U_{\sigma,j}^{(-)}(S_{\tau,j}^{(-)} + T_{\tau,j}^{(-)}) + U_{\tau,j}^{(-)}(S_{\sigma,j}^{(-)} + T_{\sigma,j}^{(-)})\right)I_{au}, \end{aligned} \quad (21)$$

where $a = \cos(\lambda)$, $b = \sin(\lambda)$. In deriving equation (21), use has been made of the following identities

$$\begin{aligned}
& (a-b)[\mathcal{L}_j, \hat{C}_{\sigma,j} \hat{C}_{\tau,j}] + (a+b)[\mathcal{L}_j, C_{\sigma,au} C_{\tau,au}] \\
&= -4ab I_{au} \left(U_{\sigma,j}^{(-)} (S_{\tau,j}^{(-)} + T_{\tau,j}^{(-)}) + U_{\tau,j}^{(-)} (S_{\sigma,j}^{(-)} + T_{\sigma,j}^{(-)}) \right) I_{au}, \\
I_{au} Q_j I_{au}^{-1} &= -\frac{i}{a} \sinh(2h) \left(U_{\sigma,j}^{(-)} C_{\tau,au} + U_{\tau,j}^{(-)} C_{\sigma,au} \right) - \frac{2i}{a} \sinh^2(h) \left(U_{\sigma,j}^{(+)} + U_{\tau,j}^{(+)} \right),
\end{aligned} \tag{22}$$

which are straightforward consequences of the identities (6). In expression (21) let us choose the parameter h by

$$\sinh(2h) = \frac{Un^2}{4} 2ab = \frac{Un^2}{4} \sin(2\lambda), \tag{23}$$

so that the second line in the r.h.s. of (21) vanish. Then the Lax-pair can be written as

$$\frac{d\mathcal{L}_j}{dt} = \mathcal{B}_{j+1} \mathcal{L}_j - \mathcal{L}_j \mathcal{B}_j, \tag{24}$$

$$\mathcal{B}_j = \mathcal{M}_j + Q_j + i \frac{Un^2}{4} \left\{ 2b \left(b - \frac{1}{a-b} \right) + \frac{a+b}{a-b} \right\} C_{\sigma,au} C_{\tau,au}. \tag{25}$$

Here, \mathcal{M}_j, Q_j are defined by equations (15) and (19), respectively. It is clear that the condition (23) guarantees the parametrizability for $SU(n)$ Hubbard model, which was first introduced in [9] through the Yang-Baxter equation. As we know, the standard method to construct the open system is to study the reflection equations [11]. The R matrix must enjoy unitarity and crossing symmetry. For $SU(n)$ Hubbard model, the R matrix in [9] does not have the crossing symmetry and the invertibility of a related matrix is lacking. In the Lax pair formalism, however, the detailed properties of the R matrix do not come in. This is another way to study the integrability of the open boundary systems. Therefore, the Lax-pair derived in the present letter would be a useful tool for analyzing open systems. We will consider its applications elsewhere.

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